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CERTAIN CONDITIONS FOR MEAN CONVERGENCE OF HERMITE AND HERMITE-FEJÉR INTERPOLATION POLYNOMIALS WITH ERDÖS-TYPE WEIGHTS

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Abstract. Let $Q \in C^1(\mathbb{R}) : \mathbb{R} \to [0, \infty)$ be an even function. We consider the exponential weight $w(x) = \exp(-Q(x))$, $x \in \mathbb{R}$. In this paper we obtain certain conditions for mean convergence theorems of the higher order Hermite-Fejér interpolation polynomials based at the zeros of the orthogonal polynomials with respect to the exponential weight.

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1. Introduction and Theorems

Consider the weight $w(x) = \exp(-Q(x))$, where $Q \in C^1(\mathbb{R}) : \mathbb{R} \to [0, \infty)$ is an even function. Suppose that $\int_0^\infty x^n w^2(x)dx < \infty$ for all $n = 0, 1, 2, \ldots$. Then we can construct the orthonormal polynomials $p_n(x) = p_n(w^2; x)$ of degree n for $w^2(x)$, that is,

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)w^2(x)dx = \delta_{mn} \quad \text{(Kronecker delta)},$$

where $p_n(x) = \gamma_n x^n + \ldots$, $\gamma_n > 0$, and the zeros of $p_n(x)$ by

$$-\infty < x_{n,n} < x_{n-1,n} < \ldots < x_{2,n} < x_{1,n} < \infty.$$

Let ℓ be a non-negative integer, and let $\ell \leq \nu - 1$. For $f \in \mathrm{C}^{(\ell)}(\mathbb{R})$ we define the (ℓ, ν)-order Hermite-Fejér interpolation polynomials $L_n(\ell, \nu, f; x) \in \mathcal{P}_{\nu n-1}$ as follows: For each $k = 1, 2, \ldots, n$,

$$L_n^{(j)}(\ell, \nu, f; x_{k,n}) = f^{(j)}(x_{k,n}), \quad j = 0, 1, 2, \ldots, \ell,$$
$$L_n^{(j)}(\ell, \nu, f; x_{k,n}) = 0, \quad j = \ell+1, \ell+2, \ldots, \nu-1,$$

where we denote $L_n^{(0)} = L_n$, $f^{(0)} = f$ and $\mathcal{P}_n$ is the class of polynomials with degree at most n. Then for each $P \in \mathcal{P}_{\nu n-1}$ we see $L_n(\nu-1, \nu, P; x) = P(x)$.

In [7], it was investigated for uniform convergence and divergence theorems of the higher order Hermite-Fejér interpolation polynomial based at the zeros of a sequence of orthogonal polynomials for polynomial exponential-type weights. Moreover, for even integers ν, [8] showed L_p-convergence theorems with respect to $L_n(\ell, \nu, f; x)$ for the zeros of a sequence of orthogonal polynomials for polynomial exponential-type weights. In this paper we will obtain certain conditions for mean convergence theorems with respect to Hermite-Fejér interpolation polynomials $L_n(\ell, \nu, f; x)$, $0 \leq \ell \leq \nu - 1$, where ν is a positive integer, based at the zeros of $p_n(w^2; x)$.

The fundamental polynomials $h_{s,k,n}(\ell,\nu;x) \in \mathcal{P}_{\nu n-1}$, $k=1,2,\ldots,n$, of $L_n(\ell,\nu,f;x)$ are defined by

$$h_{s,k,n}(\nu;x) = l_{k,n}^{\nu}(x)\sum_{i=s}^{\nu-1} e_{s,i}(\ell,\nu,k,n)(x-x_{k,n})^i,$$

satisfying $h_{s,k,n}^{(j)}(\nu;x_{p,n}) = \delta_{s,j}\delta_{k,p}$, $j,s=0,1,\ldots,\nu-1$, $p=1,2,\ldots,n$. Here, we denote $l_{k,n}(x) = \frac{p_n(w_\rho^2;x)}{(x-x_{k,n})p_n'(w_\rho^2;x_{k,n})}$. Then the (ℓ,ν)-order Hermite-Fejér interpolation polynomials $L_n(\ell,\nu,f;x)$ can be expressed as

$$L_n(\ell,\nu,f;x) = \sum_{k=1}^{n}\sum_{s=0}^{\ell} f^{(s)}(x_{k,n})h_{s,k,n}(\ell,\nu;x).$$

When $\ell=\nu-1$, for $f \in C^{(\nu-1)}(\mathbb{R})$ we have

$$L_n^{(j)}(\nu-1,\nu,f;x_{k,n}) = f^{(j)}(x_{k,n}), \quad j=0,1,\ldots,\nu-1.$$

In this paper we will give some mean convergence theorems for $L_n(\ell,\nu,f;x)$. To do this, we need some fundamental definitions. We say that $f:\mathbb{R}\to\mathbb{R}^+$ is quasi-increasing if there exists $C>0$ such that $f(x) \le Cf(y), 0<x<y$. First we need the following definition from [9].

Definition 1.1. *Let $Q:\mathbb{R}\to\mathbb{R}^+$ be a continuous even function satisfying the following properties:*
(a) *$Q'(x)$ is continuous in $\mathbb{R}$ and $Q(0)=0$.*
(b) *$Q''(x)$ exists and is positive in $\mathbb{R}\setminus\{0\}$.*
(c) $\lim_{x\to\infty} Q(x) = \infty$.
(d) *The function*

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0 \tag{1.1}$$

is quasi-increasing in $(0,\infty)$, with

$$T(x) \geqslant \Lambda > 1, \quad x \in \mathbb{R}^+\setminus\{0\}.$$

(e) *There exists $C_1>0$ such that*

$$\frac{Q''(x)}{|Q'(x)|} \le C_1\frac{|Q'(x)|}{Q(x)}, \quad a.e. \quad x \in \mathbb{R}\setminus\{0\}.$$

Then we say that $w=\exp(-Q)$ is in the class $\mathcal{F}(C^2)$. Besides, if there exists a compact subinterval $J(\ni 0)$ of $\mathbb{R}$ and $C_2>0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \geqslant C_2\frac{|Q'(x)|}{Q(x)}, \quad a.e. \quad x \in \mathbb{R}\setminus J,$$

then we say that $w=\exp(-Q)$ is in the class $\mathcal{F}(C^2+)$.

Example 1.2. We can give some typical examples in $\mathcal{F}(C^2+)$:
(1) If an exponential $Q(x)$ satisfies

$$1 < \Lambda_1 \le \frac{(xQ'(x))'}{Q'(x)} \le \Lambda_2,$$

where $\Lambda_i, i=1,2$ are constants, then we call $w(x)=\exp(-Q(x))$ the Freud weight. The class $\mathcal{F}(C^2+)$ contains the Freud weights.

(2) ([9]) For $\alpha > 1$, and a nonnegative integer r, we define

$$Q(x) = Q_{r,\alpha}(x) = \exp_r(|x|^\alpha) - \exp_r(0),$$

where if $r \geqslant 1$, then $\exp_r(x) = \exp(\exp(\exp \ldots \exp x) \ldots)(r\text{times})$, and for $r = 0$ we define $\exp_0(x) := x$.
(3) ([3]) Let us define

$$Q_{r,\alpha,m}(x) = |x|^m\{\exp_r(|x|^\alpha) - \alpha^* \exp_r(0)\}, \quad \alpha + m > 1, m \geqslant 0, \alpha \geqslant 0,$$

where for $r > 0$ we suppose $\alpha^* = 0$ if $\alpha = 0$ and otherwise $\alpha^* = 1$, and for $r = 0$ we suppose $m > 1, \alpha = 0$.
(4) ([3]) We define

$$Q_\alpha(x) = (1 + |x|)^{|x|^\alpha} - 1, \alpha > 1.$$

If $T(x)$ is bounded (unbounded), then we call the weights $w(x) = \exp(-Q(x))$ the Freud-type weights (the Erdös-type weights).

We use the following notations.
(1) Mhaskar-Rahmanov-Saff numbers a_x;

$$x = \frac{2}{\pi}\int_0^1 \frac{a_x u Q'(a_x u)}{(1-u^2)^{\frac{1}{2}}} du, \quad x > 0.$$

(2)

$$(1.2) \quad \varphi_u(x) = \begin{cases} \frac{a_u}{u} \frac{1 - \frac{|x|}{a_{2u}}}{\sqrt{1 - \frac{|x|}{a_u} + \delta_u}}, & |x| \leq a_u; \\ \varphi_u(a_u), & a_u < |x|, \end{cases} \quad \text{where} \quad \delta_u = (uT(a_u))^{-2/3}, \quad u > 0.$$

(3)

$$\Phi_n(x) := \max\left\{1 - \frac{|x|}{a_n}, \delta_n\right\} \quad \text{and} \quad \Phi(x) := \frac{1}{(1 + Q(x))^{2/3}T(x)}.$$

By [6, Lemma 3.2] we have the following:

$$\Phi(x) \leq C\Phi_n(x), \quad n \geqslant 1.$$

Lemma 1.3 ([5, Theorem 1.6]). (1) *Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$, and let $T(x)$ be unbounded. Then, for any $\eta > 0$ there exists $C(\eta) > 0$ such that*

$$a_t \leq C(\eta)t^\eta, \quad t \geqslant 1.$$

(2) *Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$, and assume*

$$(1.3) \qquad \frac{Q''(x)}{Q'(x)} \leq \lambda(b)\frac{Q'(x)}{Q(x)}, \quad |x| \geqslant b > 0,$$

where $b > 0$ is large enough. Suppose that there exist constants $\eta > 0$ and $C_1 > 0$ such that $a_t \leq C_1 t^\eta$, and if $\lambda := \lambda(b) > 1$, then there exist constants $C(\lambda, \eta)$ and $C_2 > 0$ such that for $a_t \geqslant 1$,

$$T(a_t) \leq C(\lambda, \eta)t^{\frac{2(\eta+\lambda-1)}{\lambda+1}}.$$

If $0 < \lambda \leq 1$, then for any $\mu > 0$ there exists $C(\lambda, \mu)$ such that

$$T(a_t) \leq C(\lambda, \mu)t^\mu, \quad t \geqslant 1.$$

In [6], for the Lagrange interpolation polynomial $L_n(0, 1, f; x)$ we obtained the following theorem.

Theorem 1.4 ([6, Theorem 2.2])**.** *Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$, and let $T(x)$ be unbounded. Let $1 < p < \infty$ and $\gamma > 0$, and let $\lambda = \lambda(b) \geqslant 1$ as (1.3). We suppose that for $f \in C(\mathbb{R})$*

$$\lim_{|x|\to\infty} (1+|x|)^\gamma w(x) f(x) = 0,$$

and

$$\Delta > \frac{3}{4}\frac{\lambda - 1}{\lambda - \frac{1}{3}}.$$

Then we have

$$\lim_{n\to\infty} \left\| (f - L_n(0,1,f)) w \Phi^{\Delta + \left(\frac{1}{4} - \frac{1}{p}\right)^+} \right\|_{L_p(\mathbb{R})} = 0,$$

where $x^+ = x$ if $x > 0$, otherwise $x^+ = 0$.

In this paper we will consider an analogy of Theorem 1.4 to the Hermite or Hermite-Fejér interpolation polynomial $L_n(\ell, \nu, f; x)$, $0 \le \ell \le \nu - 1$ in $L_p(\mathbb{R})$-space, $1 < p < \infty$.

To do it we need to consider a subclass of $\mathcal{F}(C^2+)$ as follows:

Definition 1.5. *Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$. Let the exponent Q satisfy that for $|x| \ge K > 0$ large enough, $Q \in C^3(\mathbb{R} \setminus \{0\})$ and*

$$\left| \frac{Q^{(3)}(x)}{Q''(x)} \right| \le C \left| \frac{Q''(x)}{Q'(x)} \right|, \tag{1.4}$$

and that for some $0 < \tau < 3/2$, $C > 0$, then for $|x| \ge K > 0$ large enough,

$$\frac{|Q'(x)|}{Q^\tau(x)} \le C. \tag{1.5}$$

Then we write $w \in \mathcal{F}_\tau(C^3+)$.

Remark 1.6. (1) *In Example 1.2, all weights satisfy all conditions of Definition 1.5.*
(2) *More generally, we can give the example of the weight $w \in \mathcal{F}_\tau(C^3+)$. Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$, and let us define*

$$\mu_+ := \limsup_{x\to\infty} \frac{Q''(x)}{Q'(x)} \Big/ \frac{Q'(x)}{Q(x)}, \quad \mu_- := \liminf_{x\to\infty} \frac{Q''(x)}{Q'(x)} \Big/ \frac{Q'(x)}{Q(x)}.$$

If $\mu_+ = \mu_-$, then we say that the weight w is regular. If $Q \in C^3(\mathbb{R} \setminus \{0\})$ satisfies (1.4), then for the regular weights we have $w \in \exp(-Q) \in \mathcal{F}_\tau(C^3+)$, that is, (1.5) holds (see [11, Corollary 5.5 (5.8)].

We have an interesting theorem as follows:

Theorem 1.7 ([11, Theorem 4.2, Theorem 4.1 (see the proofs of them)])**.** *Let r be a positive integer. Let $0 < \tau < 3/2$ and let $w = \exp(-Q) \in \mathcal{F}_\tau(C^3+)$. Then for any μ, ν, α, $\beta \in \mathbb{R}$, we can find a new weight $w_{\mu,\nu,\alpha,\beta} \in \mathcal{F}(C^2+)$ such that on $\mathbb{R}$*

$$T_w^\mu(x)(1+x^2)^\nu (1+Q(x))^\alpha (1+|Q'(x)|)^\beta w(x) \sim w_{\mu,\nu,\alpha,\beta} =: \exp(-Q_{\mu,\nu,\alpha,\beta}(x))$$

and

$$T_{w_{\mu,\nu,\alpha,\beta}}(x) \sim T_w(x), \quad a_{n/c}(w) \le a_n(w_{\mu,\nu,\alpha,\beta}) \le a_{cn}(w) (c \ge 1).$$

Here, T_w and $a_n(w)$ denote T defined in (1.1) and the Mhaskar-Rahmanov-Saff number a_n with respect to the weight w, respectively.

In the rest of this paper we write $p_n(x) := p_n(w^2, x)$. For any two sequences $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ of nonzero real numbers we write $c_n \sim d_n$ if there exists a constant $C \geq 1$ independent of n such that $\frac{1}{C}d_n \leq c_n \leq Cd_n$ for n large enough. Similarly, we define $f(x) \sim g(x)$ for two positive functions $f(x)$ and $g(x)$.

Throughout this paper, $C, C_1, C_2, ...$ denote positive constants independent of n, x, t or polynomials $P_n(x)$. The same symbol does not necessarily denote the same constant in different occurrences.

2. Theorems and Lemmas

In this paper we suppose the following.

Assumption 2.1. *Let $w(x) = \exp(-Q(x)) \in \mathcal{F}_\tau(C^3+)$, and let $T(x)$ be unbounded (recall that $w(x)$ is called an Erdös-type weight). Furthermore, we suppose that $\lambda = \lambda(b) \geqslant 1$ in (1.3), and $0 \leq \ell \leq \nu - 1$ and $\nu \geqslant 2$.*

Theorem 2.2. *Let Assumption 2.1 be satisfied. Let $1 < p < \infty$ and $\Delta > \frac{3}{4}\frac{\lambda-1}{\lambda-\frac{1}{3}}$. Assume that $u : \mathbb{R} \to [0, \infty)$ is a quasi-decreasing even function such that*

$$\left\|T^{\frac{\nu}{2}}u\right\|_{L_p(\mathbb{R})} \leq C \quad \text{and} \quad 0 < \Phi_n^{\frac{1}{4}}(x)u(x) \leq C\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}(x). \tag{2.1}$$

Then for every continuous function $f : \mathbb{R} \to \mathbb{R}$, $f \in C^{(\ell)}(\mathbb{R})$ and $\gamma > 0$ satisfying

$$\lim_{|x|\to\infty} |f^{(\ell)}(x)w^\nu(x)\Phi^{-\frac{\nu}{4}}(x)T^{\frac{\nu}{2}}(x)(1 + |x|)^{\ell+\gamma}| = 0, \tag{2.2}$$

we have

$$\lim_{n\to\infty} \|(f - L_n(\ell, \nu, f))\Phi_n^{\frac{\nu}{4}}w^\nu u\|_{L_p(\mathbb{R})} = 0. \tag{2.3}$$

Corollary 2.3. *Let Assumption 2.1 be satisfied. Let $1 < p < \infty$, and let*

$$u(x) = \Phi^{\Delta+\left(\frac{1}{4}-\frac{1}{p}\right)^+}(x) \tag{2.4}$$

and

$$\Delta > \max\left\{\frac{3}{4}\frac{\lambda-1}{\lambda-\frac{1}{3}}, \left(\frac{\nu}{2}-\frac{1}{p}\right)\frac{\lambda-1}{\lambda-\frac{1}{3}} - \left(\frac{1}{4}-\frac{1}{p}\right)^+\right\}.$$

Then under the same conditions as Theorem 2.2, (2.3) holds for every continuous function $f \in C^{(\nu-1)}(\mathbb{R})$ satisfying (2.2).

Theorem 2.4. *Let Assumption 2.1 be satisfied. Assume that for every $\varepsilon > 0$*

$$T(x) = O\left(Q^\varepsilon(x)\right). \tag{2.5}$$

Let $1 < p < 4$ and $\gamma > 1/p$ in (2.2). Let $\Delta > 1/p$ and

$$u(x) = T^{-\frac{\nu}{2}}(x)(1 + |x|)^{-\Delta}.$$

Then (2.3) holds for every continuous function $f \in C^{(\nu-1)}(\mathbb{R})$ satisfying (2.2), respectively.

Remark 2.5. (1) *To prove Theorem 2.4 we use* [1, Theorem 3.1] *with the conditions (2.5) and $1 < p < 4$.*
(2) *The condition (2.5) means $T(a_t) = O(t^\varepsilon)$, (see Lemma 2.7 (2)).*
(3) *All of four examples in Example 1.2 satisfies (2.5).*

The following lemmas are fundamental. In the rest of the paper, we often use them without notice.

Lemma 2.6. *Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$.*
(1) [9, Lemma 3.5 (3.27)-(3.29)] *For fixed $L > 0$ and uniformly for $t > 0$,*

$$a_{Lt} \sim a_t \quad \text{and} \quad T(a_{Lt}) \sim T(a_t).$$

(2) [9, Lemma 3.4 (3.18),(3.17), Lemma 3.8 (3.42)] *For $t > 0$,*

$$Q(a_t) \sim \frac{t}{\sqrt{T(a_t)}} \quad \text{and} \quad Q'(a_t) \sim \frac{t\sqrt{T(a_t)}}{a_t}.$$

(3) [9, Lemma 3.11 (a),(b)] *Given fixed $0 < \alpha$, we have, uniformly for $t > 0$,*

$$\left|1 - \frac{a_{\alpha t}}{a_t}\right| \sim \frac{1}{T(a_t)}$$

and for $0 < \alpha < 1$ there exists $C > 0$ such that for $s > 0$,

$$T(x)\left(1 - \frac{x}{a_s}\right) \geq C, \quad x \in [0, a_{\alpha s}].$$

(4) [9, Theorem 3.10 (3.47)] *Uniformly for $t > 0$,*

$$\frac{a_t'}{a_t} \sim \frac{1}{tT(a_t)}.$$

(5) [9, Lemma 3.7] *There exists $C > 0$ such that for some $0 < \varepsilon \leq 2$, and for t large enough,*

$$T(a_t) \leq Ct^{2-\varepsilon}.$$

Lemma 2.7. *Let $w \in \mathcal{F}(C^2+)$.*
(1) [9, Theorem 1.19 (e), (f)] *Uniformly for $n \geqslant 1$, $1 \leq k \leq n-1$,*

$$x_{k,n} - x_{k+1,n} \sim \varphi_n(x_{k,n})$$

and there exists $n_0 > 0$ such that for $n \geq n_0$,

$$1 - x_{1,n}/a_n \sim \delta_n.$$

Moreover,

$$\varphi_n(x_{k,n}) \sim \varphi_n(x_{k+1,n}), \quad k = 1, 2, ..., n-1.$$

(2) [6, Lemma 3.4 (d)] *Let $0 < \beta < \alpha < 1$. Let $\max\{|x_{k,n}|, |x_{k+1,n}|\} \leq a_{\alpha n}$. Then we have for $x_{k+1,n} \leq x \leq x_{k,n}$,*

$$w(x_{k,n}) \sim w(x_{k+1,n}) \sim w(x).$$

So, for given $C > 0$ and $|x| \leq a_{\beta n}$, if $|x - x_{k,n}| \leq C\varphi_n(x)$, then we have

$$w(x) \sim w(x_{k,n}).$$

Lemma 2.8. *Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$.*
(1) ([9, Theorem 12.1]) *Uniformly for $n \geqslant 1$ we have*

$$\sup_{x \in \mathbb{R}} |p_n(x)w(x)||x^2 - a_n^2|^{\frac{1}{4}} \sim 1.$$

(2) ([9, (13.11)]) *Uniformly for* $n \geqslant 1$, $1 \leq j \leq n$ *and* $x \in [x_{j+1,n}, x_{j,n}]$,

$$|(p_n w)(x)| \sim \min\{|x - x_{j,n}|, |x - x_{j+1,n}|\}\varphi_n(x_{j,n})^{-1}\left[|x_{j,n} - a_n||x_{j,n} + a_n|\right]^{-\frac{1}{4}}.$$

([9, (13.5),(13.8)]) *For* $x \in [x_{1,n}, a_n]$,

$$|p_n(x)|w(x) \leq C(x - x_{1,n})\varphi_n^{-1}(x_{1,n})[|x_{1,n} - a_n||x_{1,n} + a_n|]^{-\frac{1}{4}},$$

and for $x \in [-a_n, x_{n,n}]$,

$$|p_n(x)|w(x) \leq C(x_{n,n} - x)\varphi_n(x_{n,n})^{-1}[|x_{n,n} - a_n||x_{n,n} + a_n|]^{-\frac{1}{4}}.$$

(3) ([9, Theorem 1.19 (a)]) *Uniformly for* $n \geqslant 1$, $1 \leq j \leq n$,

$$|(p_n' w)(x_{j,n})| \sim \varphi_n^{-1}(x_{j,n})[|x_{j,n} - a_n||x_{j,n} + a_n|]^{-\frac{1}{4}}.$$

Therefore we have

$$|(p_n' w)(x_{j,n})| \sim \frac{n}{a_n^{3/2}} \frac{\left(1 - \frac{|x_{j,n}|}{a_n}\right)^{\frac{1}{4}}}{1 - \frac{|x_{j,n}|}{a_{2n}}} \tag{2.6}$$

and

$$\max_{\mathbb{R}} \left|l_{k,n}(x)w(x)w^{-1}(x_{k,n})\right| \sim 1. \tag{2.7}$$

(4) ([9, (9.28) and (13.10)]) *For* $n \geqslant 1$ *and* $1 \leq j \leq n-1$,

$$\lambda_{k,n} w^{-2}(x_{k,n}) \sim \varphi_n(x_{k,n}) \sim x_{k,n} - x_{k+1,n},$$

where $\lambda_{k,n}$ *is the Christoffel number.*

Lemma 2.9. *For a certain constant* $C > 0$,

$$\varphi_n(x) \geq C\frac{a_n}{n}\frac{1}{\sqrt{T(x)}}.$$

Proof. If $x \geq a_n$, then by (1.2) and Lemma 2.6 (3), (5) there exists $\varepsilon > 0$ such that

$$\varphi_n(x)T^{\frac{1}{2}}(x) \geq \varphi_n(a_n)T^{\frac{1}{2}}(a_n) \geq C\frac{a_n}{n}n^{\varepsilon} \geq C\frac{a_n}{n}.$$

If $a_{n/2} \leq x < a_n$, then

$$\varphi_n(x) = \frac{a_n}{n}\frac{1 - \frac{x}{a_{2n}}}{\sqrt{1 - \frac{x}{a_n} + \delta_n}} \geq \frac{a_n}{n}\frac{1 - \frac{a_n}{a_{2n}}}{\sqrt{1 - \frac{a_{n/2}}{a_n} + \delta_n}} \geq C\frac{a_n}{n}T^{-\frac{1}{2}}(a_n) \geq C_1\frac{a_n}{n}T^{-\frac{1}{2}}(x).$$

If $0 \leq x < a_{n/2}$, then

$$\varphi_n(x) = \frac{a_n}{n}\frac{1 - \frac{x}{a_{2n}}}{\sqrt{1 - \frac{x}{a_n} + \delta_n}} \geq C\frac{a_n}{n}\sqrt{1 - \frac{x}{a_n}}$$

because

$$1 - \frac{x}{a_n} \geq C\delta_n \quad \text{and} \quad 1 - \frac{x}{a_{2n}} \geq 1 - \frac{x}{a_n}.$$

On the other hand, since we see that

$$x = a_t, \quad 0 < t < n/2 \Longrightarrow a_{2t} < a_n,$$

we have

$$1 - \frac{x}{a_n} \geq 1 - \frac{x}{a_{2t}} = 1 - \frac{a_t}{a_{2t}} \sim \frac{1}{T(a_t)} = \frac{1}{T(x)}.$$

Therefore, we finally have for $0 \le x < a_{n/2}$,

$$\varphi_n(x) \ge C\frac{a_n}{n}T^{-\frac{1}{2}}(x).$$

$\square$

Lemma 2.10 ([4, Theorem 2.6]). *Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$. We have the following: For each $s = 0, 1, ..., \nu - 1$ and $i = s, s + 1, ..., \nu - 1$*

$$e_{0,0}(\ell,\nu,k,n) = 1, \quad |e_{s,i}(\ell,\nu,k,n)| \le C\left\{\frac{n}{(a_{2n}^2 - x_{k,n}^2)^{\frac{1}{2}}}\right\}^{i-s}.$$

Lemma 2.11 ([2, Theorem 2.7]). *Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$, $0 < p \le \infty$. Then for $n \ge 2$,*

$$\left\|\Phi_n^{(\frac{1}{4}-\frac{1}{p})^+} p_n w\right\|_{L_p(\mathbb{R})} \le C a_n^{\frac{1}{p}-\frac{1}{2}} \begin{cases} 1, & 0 < p < 4 \text{ or } p = \infty, \\ \log(1+n), & p \ge 4. \end{cases}$$

Lemma 2.12. *Let $w \in \mathcal{F}(C^2+)$ $0 \le \ell \le \nu - 1$. We define $x_{0,n} := (x_{1,n} + a_n)/2$ and $I_{m,n} := |x_{m,n} - x_{m-1,n}|$. If $|x| \le x_{0,n}$ and $|x - x_{m(x),n}| \le \frac{1}{2}I_{m(x),n}$, then we write $x_{m,n} := x_{m(x),n}$. If $x_{0,n} < x$, then we put $x_{m,n} = x_{m(x),n} := x_{0,n}$, and if $x < -x_{0,n}$, then we put $x_{m,n} = x_{m(x),n} := -x_{0,n}$. Then for $g \in C^{(\ell)}(\mathbb{R})$ and $0 \le \ell \le \nu - 1$, there exists a continuous function h_n satisfying*

$$h_n(x_{k,n}) = \frac{g(x_{k,n})e_{0,\nu-1}(\ell,\nu,k,n)}{\left(a_n^{\frac{1}{2}}p'_n(x_{k,n})\right)^{\nu-1}}$$

such that

(2.8)

$$\begin{aligned} |L_n(\ell,\nu,g;x)| &\le \left|\left(a_n^{\frac{1}{2}}p_n(x)\right)^{\nu-1} L_n(0,1,h_n;x)\right| \\ &+C_1\left|a_n^{\frac{1}{2}}p_n(x)\right|^{\nu}\sum_{k\ne m}\left|\left(gw^{\nu}\Phi_n^{-\frac{\nu}{4}}T^{\frac{\nu}{2}}\right)(x_{k,n})\right|\frac{T^{\frac{\nu}{2}}(x)}{|k-m|^2} \\ &+C_2\frac{a_n}{n}\left|a_n^{\frac{1}{2}}p_n(x)\right|^{\nu}\sum_{k\ne m}\sum_{s=1}^{\ell}\left|\left(g^{(s)}w^{\nu}\Phi_n^{-\frac{\nu}{4}}T^{\frac{\nu-1}{2}}\right)(x_{k,n})\right|\frac{T^{\frac{\nu-1}{2}}(x)}{|k-m|} \\ &+C_3w^{-\nu}(x)\Phi_n^{-\frac{\nu}{4}}(x)\sum_{s=0}^{\ell}\left(\frac{a_n}{n}\right)^s\left|\left(g^{(s)}w^{\nu}\Phi_n^{-\frac{\nu}{4}}\right)(x_{m,n})\right|. \end{aligned}$$

Proof. We split $L_n(\ell,\nu,g;x)$ into two parts as follows:

$$L_n(\ell,\nu,g;x) = \sum_{k=1}^{n}\sum_{s=0}^{\ell} g^{(s)}(x_{k,n})h_{s,k,n}(\ell,\nu;x) =: \sum{}' + \sum{}'',$$

where

$$\sum{}' := \sum_{k\ne m}\sum_{s=0}^{\ell} g^{(s)}(x_{k,n})h_{s,k,n}(\ell,\nu;x)$$

and

$$\sum{}'' := \sum_{k=m}\sum_{s=0}^{\ell} g^{(s)}(x_{k,n})h_{s,k,n}(\ell,\nu;x) = \sum_{s=0}^{\ell} g^{(s)}(x_{m,n})h_{s,m,n}(\ell,\nu;x).$$

Then we split $\sum'$ into three terms as follows:

$$\begin{aligned}\sum{}' &= \sum_{k\neq m}\sum_{s=0}^{\ell}\sum_{i=s}^{\nu-1} g^{(s)}(x_{k,n})l_{k,n}^{\nu}(x)e_{s,i}(\ell,\nu,k,n)(x-x_{k,n})^i \\ &= \sum_{k\neq m} g(x_{k,n})l_{k,n}^{\nu}(x)e_{0,\nu-1}(\ell,\nu,k,n)(x-x_{k,n})^{\nu-1} \\ &\quad + \sum_{k\neq m} g(x_{k,n})l_{k,n}^{\nu}(x)\sum_{i=0}^{\nu-2} e_{0,i}(\ell,\nu,k,n)(x-x_{k,n})^i \\ &\quad + \sum_{k\neq m}\sum_{s=1}^{\ell}\sum_{i=s}^{\nu-1} g^{(s)}(x_{k,n})l_{k,n}^{\nu}(x)e_{s,i}(\ell,\nu,k,n)(x-x_{k,n})^i \\ &=: A_1' + A_2' + A_3'.\end{aligned}$$

To estimate $\sum'$, we need the following estimations:

1. By Lemma 2.7 (1), for $|x-x_{k,n}| > \frac{1}{2}I_{m,n}$

$$\tag{2.9} \begin{aligned}\tfrac{1}{|x-x_{k,n}|} \leq \tfrac{1}{\sum_{m\lesseqgtr s\lesseqgtr k}\varphi_n(x_{s,n})} &\leq C\frac{n}{a_n}\frac{\max\left\{T^{\frac{1}{2}}(x), T^{\frac{1}{2}}(x_{k,n})\right\}}{|k-m|} \\ &\leq C\frac{n}{a_n}\frac{T^{\frac{1}{2}}(x)T^{\frac{1}{2}}(x_{k,n})}{|k-m|}.\end{aligned}$$

2. From (1.2) and (2.5) we see

$$\varphi_n(x_{k,n}) \sim \frac{a_n}{n}\frac{1-\frac{|x_{k,n}|}{a_{2n}}}{\sqrt{1-\frac{|x_{k,n}|}{a_n}}}.$$

3. By (2.6),

$$\tag{2.10} \begin{aligned}\tfrac{1}{|p_n'w(x_{k,n})|} &\sim \varphi_n(x_{k,n})[|x_{k,n}-a_n||x_{k,n}+a_n|]^{\frac{1}{4}} \\ &\sim \frac{a_n^{3/2}}{n}\left|1-\frac{|x_{k,n}|}{a_{2n}}\right|\left|1-\frac{|x_{k,n}|}{a_n}\right|^{-\frac{1}{4}}.\end{aligned}$$

4. By Lemma 2.10,

$$\tag{2.11} |e_{s,i}(\ell,\nu,k,n)| \leq C\left(\frac{n}{a_n}\right)^{i-s}\left(1-\frac{|x_{k,n}|}{a_{2n}}\right)^{(s-i)/2}.$$

5. By (2.9) and (2.10) we have for $0 \leq i \leq \nu-1$

$$\tag{2.12} \begin{aligned}\left|l_{k,n}^{\nu}(x)(x-x_{k,n})^i\right| &\leq C\left|p_n^{\nu}(x)w^{\nu}(x_{k,n})\right|a_n^{\frac{\nu}{2}}\left|1-\frac{|x_{k,n}|}{a_{2n}}\right|^{\nu}\left|1-\frac{|x_{k,n}|}{a_n}\right|^{-\frac{\nu}{4}} \\ &\quad\times\left(\frac{a_n}{n}\right)^i\left(\frac{\max\left\{T^{\frac{1}{2}}(x), T^{\frac{1}{2}}(x_{k,n})\right\}}{|k-m|}\right)^{\nu-i}.\end{aligned}$$

Then, when $s=0$ and $i=\nu-1$, we first see

$$\begin{aligned} & g(x_{k,n})l_{k,n}^{\nu}(x)e_{0,\nu-1}(\ell,\nu,k,n)(x-x_{k,n})^{\nu-1} \\ & = l_{k,n}(x)g(x_{k,n})\left(\frac{p_n(x)}{p_n'(x_{k,n})}\right)^{\nu-1}e_{0,\nu-1}(\ell,\nu,k,n) \\ & = p_n^{\nu-1}(x)l_{k,n}(x)\frac{g(x_{k,n})e_{0,\nu-1}(\ell,\nu,k,n)}{p_n'^{\nu-1}(x_{k,n})}. \end{aligned} \tag{2.13}$$

Secondly, when $s=0$ and $0\le i\le \nu-2$, we obtain from (2.11) and (2.12)

$$\begin{aligned} & \left|g(x_{k,n})l_{k,n}^{\nu}(x)e_{0,i}(\ell,\nu,k,n)(x-x_{k,n})^{i}\right| \\ & \le C\left|a_n^{\frac{\nu}{2}}p_n^{\nu}(x)\right|\left|\left(gw^{\nu}\Phi_n^{-\frac{\nu}{4}}T^{\frac{\nu}{2}}\right)(x_{k,n})\right|\frac{T^{\frac{\nu}{2}}(x)}{|k-m|^2}. \end{aligned} \tag{2.14}$$

Thirdly, when $s\ge 1$ and $s\le i\le \nu-1$, we have from (2.11) and (2.12)

$$\begin{aligned} & \left|g^{(s)}(x_{k,n})l_{k,n}^{\nu}(x)e_{s,i}(\ell,\nu,k,n)(x-x_{k,n})^{i}\right| \\ & \le C\left(\frac{a_n}{n}\right)^{s}\left|a_n^{\frac{1}{2}}p_n(x)\right|^{\nu}\left|\left(g^{(s)}w^{\nu}\Phi_n^{-\frac{\nu}{4}}T^{\frac{\nu-1}{2}}\right)(x_{k,n})\right|\frac{T^{\frac{\nu-1}{2}}(x)}{|k-m|}. \end{aligned} \tag{2.15}$$

Here, we used the following facts: For $s\ge 1$ and $s\le i\le \nu-1$,

$$\left|1-\frac{|x_{k,n}|}{a_n}\right|^{-\frac{\nu}{4}}\sim \Phi_n^{-\frac{\nu}{4}}(x_{k,n}),$$

$$T^{\frac{\nu-i}{2}}(x)\le T^{\frac{\nu-s}{2}}(x)\le T^{\frac{\nu-1}{2}}(x), \quad \text{and} \quad \frac{1}{|k-m|^{\nu-i}}\le\frac{1}{|k-m|}.$$

Then from (2.13)

$$A_1'=p_n^{\nu-1}(x)\sum_{k\ne m}l_{k,n}(x)\frac{g(x_{k,n})e_{0,\nu-1}(\ell,\nu,k,n)}{p_n'^{\nu-1}(x_{k,n})},$$

and from (2.14) we have the second term in the right hand side of (2.8);

$$|A_2'|\le C\left|a_n^{\frac{1}{2}}p_n(x)\right|^{\nu}\sum_{k\ne m}\left|\left(gw^{\nu}\Phi_n^{-\frac{\nu}{4}}T^{\frac{\nu}{2}}\right)(x_{k,n})\right|\frac{T^{\frac{\nu}{2}}(x)}{|k-m|^2}.$$

Moreover, from (2.15) we have the third term in the right hand side of (2.8);

$$|A_3'|\le C\frac{a_n}{n}\left|a_n^{\frac{1}{2}}p_n(x)\right|^{\nu}\sum_{k\ne m}\sum_{s=1}^{\ell}\left|\left(g^{(s)}w^{\nu}\Phi_n^{-\frac{\nu}{4}}T^{\frac{\nu-1}{2}}\right)(x_{k,n})\right|\frac{T^{\frac{\nu-1}{2}}(x)}{|k-m|}.$$

Next, we estimate $\sum''$.

$$\begin{aligned} \textstyle\sum'' & =\sum_{s=0}^{\ell}\sum_{i=s}^{\nu-1}g^{(s)}(x_{m,n})l_{m,n}^{\nu}(x)e_{s,i}(\ell,\nu,m,n)(x-x_{m,n})^{i} \\ & = g(x_{m,n})l_{m,n}^{\nu}(x)e_{0,\nu-1}(\ell,\nu,m,n)(x-x_{m,n})^{\nu-1} \\ & \quad +g(x_{m,n})l_{m,n}^{\nu}(x)\sum_{i=0}^{\nu-2}e_{0,i}(\ell,\nu,m,n)(x-x_{m,n})^{i} \\ & \quad +\sum_{s=1}^{\ell}\sum_{i=s}^{\nu-1}g^{(s)}(x_{m,n})l_{m,n}^{\nu}(x)e_{s,i}(\ell,\nu,m,n)(x-x_{m,n})^{i} \\ & =: A_1''+A_2''+A_3''. \end{aligned}$$

By (2.13) we see

$$A_1'' = p_n^{\nu-1}(x) l_{m,n}(x) \frac{g(x_{m,n}) e_{0,\nu-1}(\ell,\nu,m,n)}{{p'}_n^{\nu-1}(x_{m,n})}. \tag{2.16}$$

Now, by (2.13) and (2.16), we can easily see that there exists a continuous function satisfying

$$h_n(x_{k,n}) = \frac{g(x_{k,n}) e_{0,\nu-1}(\ell,\nu,k,n)}{\left(a_n^{\frac{1}{2}} p'_n(x_{k,n})\right)^{\nu-1}}$$

such that

$$\begin{aligned} A_1' + A_1'' &= \left(a_n^{\frac{1}{2}} p_n(x)\right)^{\nu-1} \sum_{k=1}^{n} l_{k,n}(x) \frac{g(x_{k,n}) e_{0,\nu-1}(\ell,\nu,k,n)}{\left(a_n^{\frac{1}{2}} p'_n(x_{k,n})\right)^{\nu-1}} \\ &= \left(a_n^{\frac{1}{2}} p_n(x)\right)^{\nu-1} L_n(0,1,h_n;x), \end{aligned}$$

that is, we have the first term in the right hand side of (2.8). On the other hand, since for $0 \le s \le i \le \nu-1$

$$\begin{aligned} &\left| e_{s,i}(\ell,\nu,m,n)(x-x_{m,n})^i \right| \\ &\le \left(\frac{n}{a_n}\right)^{i-s} \left(\left|1 - \frac{|x_{m,n}|}{a_{2n}}\right|\right)^{(s-i)/2} \left(\frac{a_n}{n} \frac{1-\frac{|x_{m,n}|}{a_{2n}}}{\sqrt{1-\frac{|x_{m,n}|}{a_n}}}\right)^i \\ &\le \left(\frac{a_n}{n}\right)^s \left(\left|1 - \frac{|x_{m,n}|}{a_{2n}}\right|\right)^{(s+i)/2} \left(1 - \frac{|x_{m,n}|}{a_n}\right)^{-i/2} \le C \left(\frac{a_n}{n}\right)^s \left(1 - \frac{|x_{m,n}|}{a_n}\right)^{-(\nu-1)/2} \\ &\sim \left(\frac{a_n}{n}\right)^s \Phi_n^{-(\nu-1)/2}(x_{m,n}) \le C \left(\frac{a_n}{n}\right)^s \Phi_n^{-\frac{\nu}{4}}(x_{m,n}) \Phi_n^{-\frac{\nu}{4}}(x), \end{aligned}$$

we can estimate for A_2'' and A_3'' as follows:

$$\begin{aligned} |A_2'' + A_3''| &\le C \sum_{s=0}^{\ell} \left(\frac{a_n}{n}\right)^s \left| g^{(s)}(x_{m,n}) l_{m,n}^{\nu}(x) \Phi_n^{-\frac{\nu}{4}}(x_{m,n}) \Phi_n^{-\frac{\nu}{4}}(x) \right| \\ &\le C w^{-\nu}(x) \Phi_n^{-\frac{\nu}{4}}(x) \sum_{s=0}^{\ell} \left(\frac{a_n}{n}\right)^s \left| g^{(s)}(x_{m,n}) w^{\nu}(x_{m,n}) \Phi_n^{-\frac{\nu}{4}}(x_{m,n}) \right|. \end{aligned}$$

This gives the fourth term in the right hand side of (2.8). Here we used (2.7). Consequently, we have (2.8). □

Lemma 2.13. *Let n be large enough. For x large enough,*

$$w^2(x) \Phi_n^{-\frac{1}{2}}(x) T(x) (1+|x|)^m, \quad m > 0$$

is decreasing.

Proof. First, for $L > 0$ large enough we set $L \le x \le a_n(1-\delta_n)$. Then from $\Phi_n(x) = 1 - x/a_n$,

$$\begin{aligned}
&\left(w^2(x)\left(1-\frac{x}{a_n}\right)^{-\frac{1}{2}}T(x)(1+x)^m\right)' \\
&= w^2(x)\left\{-2Q'(x)\left(1-\frac{x}{a_n}\right)^{-\frac{1}{2}}T(x)(1+x)^m\right\} \\
&\quad\times\left(1-\frac{1}{2xQ'(x)}-\frac{Q''(x)}{2(Q'(x))^2}-\frac{1}{4Q'(x)}\frac{1}{a_n-x}\right) \\
&\qquad + w^2(x)\left\{-\left(1-\frac{x}{a_n}\right)^{-\frac{1}{2}}T(x)(1+x)^m\right\}\left(\frac{T(x)}{x}-\frac{m}{1+x}\right) \\
&=: w^2(x)\left\{-2Q'(x)\left(1-\frac{x}{a_n}\right)^{-\frac{1}{2}}T(x)(1+x)^m\right\}A(x) \\
&\qquad + w^2(x)\left\{-\left(1-\frac{x}{a_n}\right)^{-\frac{1}{2}}T(x)(1+x)^m\right\}B(x).
\end{aligned}$$

For $L > 0$ large enough, we have

$$\frac{1}{2xQ'(x)} \le \frac{1}{4}.$$

From Definition 1.1, we see

$$\frac{Q''(x)}{2(Q'(x))^2} \le \frac{C}{Q(x)} \le \frac{1}{4}.$$

For $L \le x \le a_n(1-\delta_n)$ we estimate

$$C(x) := \frac{1}{4Q'(x)(a_n - x)}.$$

If $x = a_t \le a_{n/2}$, then from Lemma 2.6 (2), (3),

$$C(x) \le C\frac{a_t}{4t\sqrt{T(a_t)}}\frac{1}{a_n(1-a_t/a_{2t})} \le C\frac{a_t}{4t}\frac{\sqrt{T(a_t)}}{a_n} \le \frac{1}{4}$$

(see Lemma 2.6 (5) and [11, Lemma 3.2]). Let $a_{n/2} \le x \le a_n(1-\delta_n)$. Then, from Lemma 2.6 (1), (2) and (3), we have

$$C(x) \le C\frac{a_n}{4n\sqrt{T(a_n)}}\frac{1}{a_n\delta_n} \le \frac{C}{4}\left(\frac{T(a_n)}{n^2}\right)^{1/5} \le \frac{1}{4}.$$

Therefore, we see $A(x) > 0$. Since we also see $B(x) > 0$ easily, we have the result for $L \le x \le a_n(1-\delta_n)$. Now let $x > a_n(1-\delta_n)$. Then $\Phi_n(x) = \delta_n$. Similarly to the previous case, we have for x large enough

$$\begin{aligned}
&\left(w^2(x)\delta_n^{-\frac{1}{2}}T(x)(1+x)^m\right)' \\
&= \delta_n^{-\frac{1}{2}}w^2(x)\left\{-2Q'(x)T(x)(1+x)^m\right\}\left(1-\frac{1}{2xQ'(x)}-\frac{Q''(x)}{2Q'(x)^2}\right) \\
&+\delta_n^{-\frac{1}{2}}w^2(x)\left\{-T(x)(1+x)^m\right\}\left(\frac{T(x)}{x}-\frac{m}{1+x}\right) < 0.
\end{aligned}$$

Therefore, we have the result. □

Lemma 2.14. *Let* $0 \le \ell \le \nu - 1$, *and let*

$$\lim_{|x|\to\infty} |f^{(\ell)}(x) w^{\nu}(x) \Phi^{-\frac{\nu}{4}}(x) T^{\frac{\nu}{2}}(x)(1+|x|)^{\ell+\gamma}| = 0. \tag{2.17}$$

Let $\varepsilon \in (0,1)$. *Then we can choose a polynomial* P *such that*

$$\|(f-P)^{(\ell-s)}(x) w^{\nu}(x) \Phi_n^{-\frac{\nu}{4}}(x) T^{\frac{\nu}{2}}(x)(1+|x|)^{\ell-s+\gamma}\|_{L_\infty(\mathbb{R})} < \varepsilon, \quad s = 0, 1, ..., l. \tag{2.18}$$

Proof. Let $\varepsilon_1 \in (0,1)$. From (2.17), we can choose a polynomial R such that

$$\|(f-R)^{(\ell)}(x) w^{\nu}(x) \Phi^{-\frac{\nu}{4}}(x) T^{\frac{\nu}{2}}(x)(1+|x|)^{\ell+\gamma}\|_{L_\infty(\mathbb{R})} < \varepsilon_1$$

(see Theorem 1.7 and [10, Theorem 1.6]), therefore we have

$$\|(f-R)^{(\ell)}(x) w^{\nu}(x) \Phi_n^{-\frac{\nu}{4}}(x) T^{\frac{\nu}{2}}(x)(1+|x|)^{\ell+\gamma}\|_{L_\infty(\mathbb{R})} < \varepsilon_1 \tag{2.19}$$

because of $\Phi(x) \le C\Phi_n(x)$. We set

$$P_1(x) := R(x) + \frac{C_0}{\ell-1} x^{\ell-1} + \frac{C_1}{\ell-2} x^{\ell-2} + ... + C_{\ell-1},$$

where

$$C_0 := (f-R)^{(\ell-1)}(0), \quad C_1 := (f-R)^{(\ell-2)}(0), \quad \ldots, \quad C_{\ell-1} := (f-R)(0).$$

From (2.19) we already have the result for $s = 0$. Then we have for $s = 1$

$$\begin{aligned}
&\left\|(f-P_1)^{(\ell-1)}(x) w^{\nu}(x) \Phi_n^{-\frac{\nu}{4}}(x) T^{\frac{\nu}{2}}(x)(1+|x|)^{\ell-1+\gamma}\right\|_{L_\infty(\mathbb{R})} \\
&= \left\|\int_0^x (f-P_1)^{(\ell)}(t)dt w^{\nu}(x) \Phi_n^{-\frac{\nu}{4}}(x) T^{\frac{\nu}{2}}(x)(1+|x|)^{\ell-1+\gamma}\right\|_{L_\infty(\mathbb{R})} \\
&= \|x(f-R)^{(\ell)}(\xi_x) w^{\nu}(x) \Phi_n^{-\frac{\nu}{4}}(x) T^{\frac{\nu}{2}}(x)(1+|x|)^{\ell-1+\gamma}\|_{L_\infty(\mathbb{R})} \\
&\le C\|(f-R)^{(\ell)}(\xi_x) w^{\nu}(\xi_x) \Phi_n^{-\frac{\nu}{4}}(\xi_x) T^{\frac{\nu}{2}}(\xi_x)(1+|\xi_x|)^{\ell+\gamma}\|_{L_\infty(\mathbb{R})} \le c_1\varepsilon_1
\end{aligned}$$

because we see that $w^{\nu}(x)\Phi_n^{-\frac{\nu}{4}}(x)T^{\frac{\nu}{2}}(x)(1+|x|)^{\ell+\gamma}$ is quasi-decreasing by Lemma 2.13. Inductively, there exists $P_s(x) \in \mathcal{P}_{\ell-1}$, $2 \le s \le \ell$ such that $P_s'(x) = P_{s-1}(x)$ and

$$\left\|(f-P_s)^{(\ell-s)}(x) w^{\nu}(x) \Phi_n^{-\frac{\nu}{4}}(x) T^{\frac{\nu}{2}}(x)(1+|x|)^{\ell-s+\gamma}\right\|_{L_\infty(\mathbb{R})} < c_1 \ldots c_s \varepsilon_1.$$

We may consider the above inequation as $c_i \ge 1$, $i = 1, 2, \ldots, \ell$. Then we put $c_1 \ldots c_s \varepsilon_1 = \varepsilon$, $P = P_\ell$. For these ε and P, we have the result. □

Remark 2.15. *Under the condition (2.17) we have*

$$\left\|f^{(\ell-s)}(x) w^{\nu}(x) \Phi_n^{-\frac{\nu}{4}}(x) T^{\frac{\nu}{2}}(x)(1+|x|)^{\ell-s+\gamma}\right\|_{L_\infty(\mathbb{R})} < C(f), \quad s = 0, 1, ..., l,$$

where $C(f)$ *is a constant depending on* f. *In fact, since* P *is a polynomial and* $\|wx^j\|_{L_\infty(\mathbb{R})} < \infty$, $j = 0, 1, \ldots,$ *we see*

$$\left\|P^{(\ell-s)}(x) w^{\nu}(x) \Phi_n^{-\frac{\nu}{4}}(x) T^{\frac{\nu}{2}}(x)(1+|x|)^{\ell-s+\gamma}\right\|_{L_\infty(\mathbb{R})} < C(f), \quad s = 0, 1, \ldots, \ell,$$

then, in the proof of Lemma 2.14 we may exchange ε *with* $C(f)$.

3. Proof of Theorems

Lemma 3.1. *Let $0 \le \ell \le \nu - 1$ and $\varepsilon > 0$. Then for $f \in C^{(\ell)}(\mathbb{R})$ satisfying (2.17), there exists a polynomial P which satisfies (2.18) (see Lemma 2.14). If we put $g := f - P$, then we have*

$$\left\| L_n(\ell, \nu, g; x)\Phi_n^{\frac{\nu}{4}}(x)w^{\nu}(x)u(x)\right\|_{L_p(\mathbb{R})} \le C_1 \left\| L_n(0, 1, g\beta_n; x)\Phi_n^{\frac{1}{4}}(x)w(x)u(x)\right\|_{L_p(\mathbb{R})} + C_2\varepsilon \left\| T^{\frac{\nu}{2}}(x)u(x)\right\|_{L_p(\mathbb{R})},$$

where $\beta_n(x)$ is a continuous function with

$$\beta_n(x_{k,n}) = \frac{e_{0,\nu-1}}{\{a_n^{\frac{1}{2}} p_n'(x_{k,n})\}^{\nu-1}}.$$

Proof. We use Lemma 2.11 with $p = \infty$ and Lemma 2.12. Then, using (2.8) and (2.18), we have

$$\begin{aligned}|L_n(\ell, \nu, g; x)| \le & \left| w^{-\nu+1}(x)\Phi_n^{-\frac{\nu-1}{4}}(x)L_n(0, 1, g\beta_n; x)\right| \\ & + C\varepsilon w^{-\nu}(x)\Phi_n^{-\frac{\nu}{4}}(x)\Big\{ \sum_{k\ne m} (1 + |x_{m,n}|)^{-\gamma} \frac{T^{\frac{\nu}{2}}(x)}{|k-m|^2} \\ & + \frac{a_n}{n}\sum_{k\ne m}\sum_{s=1}^{\ell-1} (1 + |x_{k,n}|)^{-s-\gamma}\frac{T^{\frac{\nu-1}{2}}(x)}{|k-m|} + \sum_{s=0}^{\ell}\left(\frac{a_n}{n}\right)^s (1 + |x_{m,n}|)^{-s-\gamma}\Big\} \\ \le & \left| w^{-\nu+1}(x)\Phi_n^{-\frac{\nu-1}{4}}(x)L_n(0, 1, g\beta_n; x)\right| + C_1\varepsilon w^{-\nu}(x)\Phi_n^{-\frac{\nu}{4}}(x)T^{\frac{\nu}{2}}(x),\end{aligned}$$

because

$$\sum_{k\ne m}\frac{1}{|k-m|^2} < \infty, \quad \frac{a_n}{n}\sum_{k\ne m}\frac{(1 + |x_{k,n}|)^{s-\ell-\gamma}}{|k-m|} < \infty$$

and

$$\left| a_n^{\frac{1}{2}} p_n w(x)\right| \le C\Phi_n^{-\frac{1}{4}}(x).$$

Hence, we have

$$\left\| L_n(\ell, \nu, g; x)\Phi_n^{\frac{\nu}{4}}(x)w^{\nu}(x)u(x)\right\|_{L_p(\mathbb{R})} \le \left\| L_n(0, 1, g\beta_n; x)\Phi_n^{\frac{1}{4}}(x)w(x)u(x)\right\|_{L_p(\mathbb{R})} + C_2\varepsilon \left\| T^{\frac{\nu}{2}}(x)u(x)\right\|_{L_p(\mathbb{R})}.$$

□

Let $\phi(x) := (1 + |x|)^{-\gamma}$, $\gamma > 0$. Then we have the following lemmas.

Lemma 3.2. [6, Lemma 4.4] *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions from $\mathbb{R} \to \mathbb{R}$ such that for $n \geqslant 1$,*

$$f_n(x) = 0, |x| < a_{\frac{n}{9}}; \quad |f_n w|(x) \le \phi(x), x \in \mathbb{R}. \tag{3.1}$$

Then, for $1 \le p \le \infty$ and $\Delta > 0$, we have

$$\lim_{n\to} \|L_n[f_n]w\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\|_{L_p(\mathbb{R})} = 0.$$

Lemma 3.3. [6, Lemma 4.5] *Let $1 \le p \le \infty$. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of measurable functions from $\mathbb{R} \to \mathbb{R}$ such that for $n \geqslant 1$,*

$$g_n(x) = 0, |x| \geqslant a_{\frac{n}{9}}; \quad |g_n w|(x) \le \phi(x), x \in \mathbb{R}. \tag{3.2}$$

Let us suppose

$$\Delta > \frac{3}{4}\frac{\lambda - 1}{\lambda - \frac{1}{3}},$$

where $\lambda = \lambda(b) \geqslant 1$ is defined in (1.3). Then for $1 \le p \le \infty$, we have

$$\lim_{n\to\infty} \|L_n[g_n] w \Phi^{\Delta + (\frac{1}{4} - \frac{1}{p})^+}\|_{L_p(|x| \geqslant a_{\frac{n}{8}})} = 0.$$

Lemma 3.4. [6, Lemma 4.7] *Let $1 < p < \infty$ and $0 < \varepsilon < 1$. We define $\{g_n\}_{n=1}^{\infty}$ as in Lemma 3.3, but we exchange (3.2) with*

$$|g_n(x) w(x)| \le \varepsilon \phi(x), \ x \in \mathbb{R}, \ n \geqslant 1. \tag{3.3}$$

Then, for $1 < p < \infty$,

$$\limsup_{n\to\infty} \|L_n[g_n] w \Phi^{\Delta + (\frac{1}{p} - \frac{1}{4})^+}\|_{L_p(|x| \le a_{\frac{n}{8}})} \le C\varepsilon.$$

Remark 3.5. *Even if we exchange $x \in \mathbb{R}$ of (3.1), (3.2) and (3.3) with $x_{k,n}$, $k = 1, 2, \ldots, n$, Lemma 3.2-3.4 are still established (see* [6, proofs of Lemma 4.4, 4.5, 4.7]*).*

To prove Theorem 2.2, we need the following lemma.

Lemma 3.6. *Let $w \in \mathcal{F}(C^2+)$. If*

$$\left| g^{(s)}(x) w^{\nu}(x) \Phi^{-\frac{\nu}{4}}(x) T^{\frac{\nu}{2}}(x) \right| < C, \quad s = \ell + 1, \ldots, \nu - 1,$$

then

$$\left| \sum_{k=1}^{n} \sum_{s=l+1}^{\nu-1} g^{(s)}(x_{k,n}) h_{s,k,n}(x) \right| \le C \left(\frac{a_n}{n}\right)^{\ell+1} w^{-\nu}(x) \Phi_n^{-\frac{\nu}{4}}(x) T^{\frac{\nu-1}{2}}(x) \log n.$$

Proof. Similarly to the estimation of A_3' and $\sum''$ in the proof of Lemma 2.12, we have from (2.15)

$$\begin{aligned}
&\left| \sum_{k\neq m} \sum_{s=l+1}^{\nu-1} g^{(s)}(x_{k,n}) h_{s,k,n}(x) \right| \\
&= \left| \sum_{k\neq m} \sum_{s=l+1}^{\nu-1} \sum_{i=s}^{\nu-1} g^{(s)}(x_{k,n}) l_{k,n}^{\nu}(x) e_{s,i}(\ell, \nu, k, n)(x - x_{k,n})^i \right| \\
&\le C \left| a_n^{\frac{1}{2}} p_n(x) \right|^{\nu} \sum_{k\neq m} \sum_{s=l+1}^{\nu-1} \sum_{i=s}^{\nu-1} \left(\frac{a_n}{n}\right)^s \left| g^{(s)}(x_{k,n}) w^{\nu}(x_{k,n}) \Phi_n^{-\frac{\nu}{4}}(x_{k,n}) \right| \\
&\qquad\qquad \times \frac{T^{\frac{\nu-1}{2}}(x_{k,n}) T^{\frac{\nu-1}{2}}(x)}{|k-m|} \\
&\le C \left(\frac{a_n}{n}\right)^{\ell+1} \left| a_n^{\frac{1}{2}} p_n(x) \right|^{\nu} T^{\frac{\nu-1}{2}}(x) \log n \le C \left(\frac{a_n}{n}\right)^{\ell+1} w^{-\nu}(x) \Phi_n^{-\frac{\nu}{4}}(x) T^{\frac{\nu-1}{2}}(x) \log n
\end{aligned}$$

by Lemma 2.11 with $p=\infty$, $t=1$ and

$$\begin{aligned}&\sum_{s=l+1}^{\nu-1}\left|g^{(s)}(x_{m,n})h_{s,m,n}(x)\right|\\&\le C\sum_{s=\ell+1}^{\nu-1}\left(\frac{a_n}{n}\right)^s\left|g^{(s)}(x_{m,n})w^{\nu}(x_{m,n})\Phi_n^{-\frac{\nu}{4}}(x_{m,n})w^{-\nu}(x)\Phi_n^{-\frac{\nu}{4}}(x)\right|\\&\le C\left(\frac{a_n}{n}\right)^{\ell+1}w^{-\nu}(x)\Phi_n^{-\frac{\nu}{4}}(x).\end{aligned}$$

□

Proof of Theorem 2.2. First, we consider the case of $\ell=\nu-1$. Let $\varepsilon>0$. Then we select a polynomial P satisfying (2.18). We see

$$f-L_n(\nu-1,\nu,f)=f-P+L_n(\nu-1,\nu,P-f).$$

Now we obtain

$$\|(f(x)-P(x))\Phi_n^{\frac{\nu}{4}}(x)w^{\nu}(x)u(x)\|_{L_p(\mathbb{R})}\le\varepsilon\|\Phi_n^{\frac{\nu}{2}}T^{-\frac{\nu}{2}}(x)u(x)\|_{L_p(\mathbb{R})}\le C\varepsilon. \tag{3.4}$$

So we have

$$\begin{aligned}&\|(f(x)-L_n(\nu-1,\nu,f;x))\Phi_n^{\frac{\nu}{4}}(x)w^{\nu}(x)u(x)\|_{L_p(\mathbb{R})}\\&\le C\varepsilon+\|L_n(\nu-1,\nu,P-f;x)\Phi_n^{\frac{\nu}{4}}(x)w^{\nu}(x)u(x)\|_{L_p(\mathbb{R})}\\&=:C\varepsilon+\int.\end{aligned} \tag{3.5}$$

Now, we estimate $\int$. By Lemma 3.1, we have

(3.6)
$$\begin{aligned}\int&=\left\|L_n(\nu-1,\nu,P-f;x)\Phi_n^{\frac{\nu}{4}}(x)w^{\nu}(x)u(x)\right\|_{L_p(\mathbb{R})}\\&\le C_1\left\|L_n(0,1,(P-f)\beta_n;x)\Phi_n^{\frac{1}{4}}(x)w(x)u(x)\right\|_{L_p(\mathbb{R})}+C_2\varepsilon\left\|T^{\frac{\nu}{2}}(x)u(x)\right\|_{L_p(\mathbb{R})}.\end{aligned}$$

Lemma 2.14 (2.18) implies

$$|f(x)-P(x)|\,w(x)<\varepsilon\Phi_n^{\frac{\nu}{4}}(x)T^{-\frac{\nu}{2}}(x)w^{-\nu+1}(x)(1+|x|)^{-\gamma}$$

and then since

$$\beta_n(x_{k,n})=\frac{e_{0,\nu-1}}{\left(a_n^{\frac{1}{2}}p_n'(x_{k,n})\right)^{\nu-1}},$$

we have

$$|(f(x_{k,n})-P(x_{k,n}))\,\beta_n(x_{k,n})w(x_{k,n})|<\varepsilon\frac{e_{0,\nu-1}\Phi_n^{\frac{\nu}{4}}(x_{k,n})T^{-\frac{\nu}{2}}(x_{k,n})(1+|x_{k,n}|)^{-\gamma}}{\left|a_n^{\frac{1}{2}}p_n'(x_{k,n})w(x_{k,n})\right|^{\nu-1}}.$$

By Lemma 2.8 (3),

$$\frac{1}{\left|a_n^{\frac{1}{2}}p_n'(x_{k,n})w(x_{k,n})\right|}\sim\varphi_n(x_{k,n})\left|1-\frac{|x_{k,n}|}{a_n}\right|^{\frac{1}{4}},$$

and using

$$|e_{0,\nu-1}(\ell,\nu,k,n)| \le C\left\{\frac{n}{(a_{2n}^2 - x_{k,n}^2)^{\frac{1}{2}}}\right\}^{\nu-1}, \tag{3.7}$$

we have

$$\begin{aligned}\frac{e_{0,\nu-1}\Phi_n^{\frac{\nu}{4}}(x_{k,n})T^{-\frac{\nu}{2}}(x_{k,n})(1+|x_{k,n}|)^{-\gamma}}{\left|a_n^{\frac{1}{2}}p_n'(x_{k,n})w(x_{k,n})\right|^{\nu-1}} &\le C\left|1-\frac{|x_{k,n}|}{a_{2n}}\right|^{\frac{\nu-1}{2}} T^{-\frac{\nu}{2}}(x_{k,n})(1+|x_{k,n}|)^{-\gamma}\\ &\le CT^{-\frac{\nu}{2}}(x_{k,n})(1+|x_{k,n}|)^{-\gamma}.\end{aligned}$$

Therefore, we know

$$|(f(x_{k,n})-P(x_{k,n}))\,\beta_n(x_{k,n})w(x_{k,n})| < C\varepsilon T^{-\frac{\nu}{2}}(x_{k,n})(1+|x_{k,n}|)^{-\gamma}.$$

Let $\chi_n := \chi[-a_{\frac{n}{9}}, a_{\frac{n}{9}}]$ and

$$(f-P)\beta_n = \{(f-P)\beta_n\}\chi_n + \{(f-P)\beta_n\}(1-\chi_n) =: g_n(x) + f_n(x).$$

Then we see that f_n and g_n satisfy (3.1), (3.2) and (3.3) for $x_{k,n}$, $k = 1, 2, \ldots, n$ instead of $x \in \mathbb{R}$. Then from Lemma 3.2-3.4 and Remark 3.5, we have

$$\lim_{n\to\infty}\left\|L_n(0,1,(f-P)\beta_n;x)w(x)\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}(x)\right\|_{L_p(\mathbb{R})} = 0. \tag{3.8}$$

Consequently, by (3.6) we have

$$0 \le \lim_{n\to\infty}\int < C_3\varepsilon,$$

under the conditions

$$\left\|T^{\frac{\nu}{2}}(x)u(x)\right\|_{L_p(\mathbb{R})} \le C \quad \text{and} \quad 0 < \Phi_n^{\frac{1}{4}}u(x) \le C\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}(x).$$

Therefore, with (3.5) we have the result. Next, we let $0 \le \ell \le \nu-2$. Let $\varepsilon > 0$. Then we select a polynomial P satisfying (2.18). We see

$$\begin{aligned} f - L_n(\ell,\nu,f) &= f - P + P - L_n(\ell,\nu,f)\\ &= f - P + L_n(\nu-1,\nu,P) - L_n(\ell,\nu,f)\\ &= (f-P) + L_n(\ell,\nu,P-f) + \sum_{k=1}^{n}\sum_{s=l+1}^{\nu-1} P^{(s)}(x_{k,n})h_{s,k,n}(x).\end{aligned} \tag{3.9}$$

The condition (2.2) implies (2.18), then we have (3.4), and furthermore by Lemma 3.1 we obtain

(3.10)

$$\begin{aligned}&\left\|L_n(\ell,\nu,P-f;x)\Phi_n^{\frac{\nu}{4}}(x)w^{\nu}(x)u(x)\right\|_{L_p(\mathbb{R})}\\ &\le C_1\left\|L_n(0,1,(P-f)\beta_n;x)\Phi_n^{\frac{1}{4}}(x)w(x)u(x)\right\|_{L_p(\mathbb{R})} + C_2\varepsilon\left\|T^{\frac{\nu}{2}}(x)u(x)\right\|_{L_p(\mathbb{R})}.\end{aligned}$$

Here, we note that (3.8) holds(see (2.1)). Now we estimate the third term in (3.9). As we note in Remark 2.15, we have for $s = 0, 1, \ldots, \nu-1$,

$$\left|P^{(s)}(x)w^{\nu}(x)\Phi_n^{-\frac{\nu}{4}}(x)T^{\frac{\nu}{2}}(x)(1+|x|)^{s+\gamma}\right| < C(f), \quad s = 0, 1, ..., l.$$

Hence, using Lemma 3.6, we have for $s = l+1, l+2, ..., \nu - 1$ and $0 < \delta < 1$,

$$\lim_{n\to\infty}\left\|\Phi_n^{\frac{\nu}{4}}(x)w^{\nu}(x)u(x)\sum_{k=1}^{n}P^{(s)}(x_{k,n})h_{s,k,n}(x)\right\|_{L_p(\mathbb{R})}$$

$$\le C\lim_{n\to\infty}\left(\frac{a_n}{n}\right)^{\ell+1}\log n\left\|u(x)T^{\frac{\nu-1}{2}}(x)\right\|_{L_p(\mathbb{R})} = 0. \tag{3.11}$$

Consequently, noting (3.9) and (3.10), we have the result (2.3) by (3.4), (3.8) and (3.11). □

4. Proofs of Corollary 2.3 and Theorem 2.4

Lemma 4.1. *Let $1 < p < \infty$, and let $\nu \geqslant 2$. If*

$$\Delta > \left(\frac{\nu}{2}-\frac{1}{p}\right)\frac{\lambda-1}{\lambda-\frac{1}{3}} - \left(\frac{1}{4}-\frac{1}{p}\right)^{+},$$

then we have

$$\|T^{\frac{\nu}{2}}(x)\Phi^{\Delta+\left(\frac{1}{4}-\frac{1}{p}\right)^{+}}(x)\|_{L_p(\mathbb{R})} = O(1). \tag{4.1}$$

Proof. Let $\mu := \Delta + \left(\frac{1}{4}-\frac{1}{p}\right)^{+}$. Let $|x| := a_t$, $t \geqslant 1$. Then, using Lemma 2.6 (2), we see

$$\left|T^{\frac{\nu}{2}}(x)\Phi^{\Delta+\left(\frac{1}{4}-\frac{1}{p}\right)^{+}}(x)\right| \le C\frac{T^{\frac{\nu}{2}-\frac{2}{3}\mu}(a_t)}{t^{\frac{2}{3}\mu}}.$$

Since $a_t' \le Ca_t/(tT(a_t))$, we see

$$\begin{aligned} L &:= \left\|T^{\frac{\nu}{2}}(x)\Phi^{\mu}(x)\right\|^{p}_{L_p(a_1\le|x|)} \le C\int_1^{\infty}\left|\frac{T^{\frac{\nu}{2}-\frac{2}{3}\mu}(a_t)}{t^{\frac{2}{3}\mu}}\right|^{p}\frac{a_t}{tT(a_t)}dt \\ &= C\int_1^{\infty}\left|\frac{T^{(\frac{\nu}{2}-\frac{2}{3}\mu)p-1}(a_t)}{t^{\frac{2}{3}\mu p+1}}\right|a_t dt. \end{aligned}$$

We use

$$T(a_t) \le C(\lambda,\eta)t^{\frac{2(\eta+\lambda-1)}{\lambda+1}}.$$

Then

$$L \le C_1(\lambda,\eta)\int_1^{\infty}\left|\frac{t^{\{(\frac{\nu}{2}-\frac{2}{3}\mu)p-1\}\frac{2(\eta+\lambda-1)}{\lambda+1}}}{t^{\frac{2}{3}\mu p+1}}\right|t^{\eta}dt.$$

We will show

$$\frac{2}{3}\mu p + 1 - \left\{\left(\frac{\nu}{2}-\frac{2}{3}\mu\right)p-1\right\}\frac{2(\eta+\lambda-1)}{\lambda+1} - \eta > 1, \tag{4.2}$$

that is,

$$K := \frac{2}{3}\mu p - \left\{\left(\frac{\nu}{2}-\frac{2}{3}\mu\right)p-1\right\}\frac{2(\eta+\lambda-1)}{\lambda+1} - \eta > 0.$$

We can prove it as follows: For $\eta > 0$ small enough,

$$\begin{aligned}
&\mu > \left(\frac{\nu}{2} - \frac{1}{p}\right) \frac{\lambda-1}{\lambda-\frac{1}{3}} \\
&\Leftrightarrow \frac{2}{3}\mu p \left(1 + \frac{2(\lambda-1)}{\lambda+1}\right) > \left(\frac{\nu p}{2} - 1\right) \frac{2(\lambda-1)}{\lambda+1} \\
&\Leftrightarrow \frac{2}{3}\mu p - \left\{\left(\frac{\nu}{2} - \frac{2}{3}\mu\right) p - 1\right\} \frac{2(\lambda-1)}{\lambda+1} > 0 \\
&\Rightarrow \frac{2}{3}\mu p - \left\{\left(\frac{\nu}{2} - \frac{2}{3}\mu\right) p - 1\right\} \frac{2(\lambda-1)}{\lambda+1} > \left[\left\{\left(\frac{\nu}{2} - \frac{2}{3}\mu\right) p - 1\right\} \frac{2}{\lambda+1} + 1\right] \eta \\
&\Rightarrow K > 0.
\end{aligned}$$

Thus we have (4.2). Then there exists $\delta > 0$ such that

$$L \leq C_1(\lambda, \eta) \int_1^\infty \frac{1}{t^{1+\delta}} dt \leq CC_1(\lambda, \eta).$$

So we conclude (4.1). □

Proof of Corollary 2.3. From Lemma 4.1 and (2.4) we see that u satisfies (2.1). Therefore, it is proven. □

Proof of Theorem 2.4. We repeat the method of the proof for Theorem 2.2. Let $0 \leq \ell \leq \nu - 1$. For $0 < \varepsilon < 1$ we select a polynomial P satisfying (2.18). From (3.9) and (3.10) we have

(4.3)
$$\begin{aligned}
&\left\|\left(f(x) - L_n(\ell, \nu, f; x)\right) \Phi_n^{\frac{\nu}{4}}(x) w^\nu(x) u(x)\right\|_{L_p(\mathbb{R})} \\
&\leq \left\|\left(f(x) - P(x)\right) \Phi_n^{\frac{\nu}{4}}(x) w^\nu(x) u(x)\right\|_{L_p(\mathbb{R})} \\
&+C_1 \left\|L_n(0, 1, (P-f)\beta_n; x) \Phi_n^{\frac{1}{4}}(x) w(x) u(x)\right\|_{L_p(\mathbb{R})} + C_2\varepsilon \left\|T^{\frac{\nu}{2}}(x) u(x)\right\|_{L_p(\mathbb{R})} \\
&+ \left\|\Phi_n^{\frac{\nu}{4}}(x) w^\nu(x) u(x) \sum_{k=1}^n \sum_{s=l+1}^{\nu-1} P^{(s)}(x_{k,n}) h_{s,k,n}(x)\right\|_{L_p(\mathbb{R})} \\
&=: A + B + C + D,
\end{aligned}$$

but if $\ell = \nu - 1$, then we omit D. From (2.18) and the definition of u, We know

$$A + C = O(\varepsilon), \tag{4.4}$$

and when $0 \leq \ell \leq \nu - 2$, as (3.11) we obtain

$$\lim_{n\to\infty} D = 0. \tag{4.5}$$

Now we estimate B. From [1, Theorem 3.1] we have

$$\begin{aligned}
&\|L_n(0, 1, (P-f)\beta_n; x) \Phi_n^{\frac{1}{4}}(x) w(x) u(x)\|_{L_p(\mathbb{R})} \\
&\leq \|L_n(0, 1, (P-f)\beta_n; x) w(x)\|_{L_p(\mathbb{R})} \\
&\leq C \left\{\sum_{k=1}^n \lambda_{k,n} w^{-2}(x_{k,n}) \left|L_n(0, 1, (P-f)\beta_n; x_{k,n}) w(x_{k,n})\right|^p\right\}^{\frac{1}{p}} \\
&\leq C \left\{\sum_{k=1}^n \lambda_{k,n} w^{-2}(x_{k,n}) \left|(P-f)(x_{k,n}) \beta_n(x_{k,n}) w(x_{k,n})\right|^p\right\}^{\frac{1}{p}}.
\end{aligned} \tag{4.6}$$

By (2.6) and (3.7) we see

$$\left|\beta_n(x_{k,n}) w^{-\nu+1}(x_{k,n}) \Phi_n^{\frac{\nu-1}{4}}(x_{k,n})\right| \leq O(1)$$

and we have from (2.18)

$$\begin{aligned}|(P-f)(x_{k,n})\beta_n(x_{k,n})w(x_{k,n})| &= \left|(P-f)(x_{k,n})w^{\nu}(x_{k,n})\beta_n(x_{k,n})w^{-\nu+1}(x_{k,n})\right| \\ &\le C\varepsilon(1+|x_{k,n}|)^{-\gamma}.\end{aligned}$$

So if we continue to estimate, using $\gamma p > 1$ and

$$\lambda_{k,n}w^{-2}(x_{k,n}) \sim \varphi_n(x_{k,n})$$

(see Lemma 2.8 (4)), we have for (4.6)

$$\begin{aligned}&\|L_n(0,1,(P-f)\beta_n;x)\Phi_n^{\frac{1}{4}}(x)w(x)u(x)\|_{L_p(\mathbb{R})} \\ &\le C\varepsilon\left\{\sum_{k=1}^{n}\varphi_n(x_{k,n})(1+|x_{k,n})^{-\gamma p}\right\}^{\frac{1}{p}} \le C\varepsilon\left\{\int_{-\infty}^{\infty}(1+|t|)^{-\gamma p}dt\right\}^{\frac{1}{p}} = O(\varepsilon).\end{aligned}$$

Hence we conclude

$$B = O(\varepsilon). \tag{4.7}$$

Consequently, by (4.3), (4.4), (4.5), (4.6), and (4.7) the proof of Theorem 2.4 is complete. □

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